MIXING ACTIONS OF HEISENBERG GROUP

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Abstract. Mixing (of all orders) rank-one actions T of Heisenberg group $H_3(\mathbb{R})$ are constructed. The restriction of T to the center of $H_3(\mathbb{R})$ is simple and commutes only with T. Mixing Poisson and mixing Gaussian actions of $H_3(\mathbb{R})$ are also constructed. A rigid weakly mixing rank-one action T is constructed such that the restriction of T to the center of $H_3(\mathbb{R})$ is not isomorphic to its inverse.

0. Introduction

We state a question by Dan Rudolph:

Problem 1. *Which amenable locally compact second countable groups G admit mixing free actions with zero entropy?*

Consider a subtler problem:

Problem 2. *Which amenable locally compact second countable groups G admit mixing rank-one free actions?*

We recall that a measure preserving action $T = (T_g)_{g\in G}$ of G on a standard probability space $(X, \mathfrak{B}, \mu)$ is

— *mixing* if $\lim_{g\to G} \mu(T_g A \cap B) = \mu(A)\mu(B)$ for all $A, B \in \mathfrak{B}$,
— *mixing of order* l if for each $\epsilon > 0$ and subsets $A_0, A_1, \dots, A_l \in \mathfrak{B}$, there is a compact subset $K \subset G$ such that

$$|\mu(T_{g_0} A_0 \cap \dots \cap T_{g_l} A_l) - \mu(A_0)\mu(A_1)\cdots\mu(A_l)| < \epsilon$$

for each collection $g_0, \dots g_l \in G$ with $g_i g_j^{-1} \notin K$,
— *rigid* if there is a sequence $g_n \to \infty$ in G such that $\lim_{g\to G} \mu(T_g A \cap B) = \mu(A \cap B)$ for all $A, B \in \mathfrak{B}$

We recall now the definition of rank one. Fix a Følner sequence $(F_n)_{n=1}^{\infty}$ in G.

— A *Rokhlin tower or column* for T is a triple (Y, f, F), where $Y \in \mathfrak{B}$, F is a relatively compact subset of G and $f : Y \to F$ is a measurable mapping such that for any Borel subset $H \subset F$ and an element $g \in G$ with $gH \subset F$, one has $f^{-1}(gH) = T_g f^{-1}(H)$.
— The action T has *rank one along* $(F_n)_{n\in\mathbb{N}}$ if there exists a sequence of Rokhlin towers (Y_n, f_n, F_n) such that for each subset $B \in \mathfrak{B}$, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n\to\infty} \mu(B \triangle f_n^{-1}(H_n)) = 0.$$

It is easy to see that each rank-one action is ergodic. It is well known that each rank-one action has zero entropy. Hence every solution of Problem 2 is a solution of Problem 1. The two problems are still open. However various constructions of mixing rank-one actions are elaborated for the following amenable groups:

— $G = \mathbb{Z}$ in [Or], [Ad], [CrSi], [Ry4], etc.
— $G = \mathbb{Z}^2$ in [AdSi],
— $G = \mathbb{R}$ in [Pr], [Fa],
— $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ for arbitrary $d_1, d_2 \geq 0$ in [DaSi1],
— $G = \bigoplus_{j=0}^{\infty} G_j$, where G_j is a finite group [Da2],
— G is a locally normal discrete countable group, i.e. $G = \bigcup_{j=1}^{\infty} G_j$ for a nested sequence $G_1 \subset G_2 \subset \cdots$ of normal finite subgroups $G_j \subset G$ [Da4].

The Heisenberg group $H_3(\mathbb{R})$ consists of of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are arbitrary reals. The Heisenberg group endowed with the natural topology is a connected, simply-connected nilpotent Lie group. We now let

$$a(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad c(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\{a(t) \mid t \in \mathbb{R}\}$, $\{b(t) \mid t \in \mathbb{R}\}$ and $\{c(t) \mid t \in \mathbb{R}\}$ are three closed one-parameter subgroups in $H_3(\mathbb{R})$. The latter one is the center of $H_3(\mathbb{R})$. Every element g of $H_3(\mathbb{R})$ can be written uniquely as the product $g = a(t_1)b(t_2)c(t_3)$ for some $t_1, t_2, t_3 \in \mathbb{R}$. We also note that $[a(t_1), b(t_2)] := a(t_1)b(t_2)a(t_1)^{-1}b(t_2)^{-1} = c(t_1 t_2)$.

We now state one of the main results of the present paper.

Theorem 0.1. *There exist mixing of all orders rank-one (and hence zero entropy) action T of Heisenberg group $H_3(\mathbb{R})$.*

We construct these actions utilizing the (C, F)-construction with randomly chosen 'spacers' in the spirit of Ornstein's rank-one mixing map [Or] (see also [Ru]). This construction is an algebraic counterpart of the well-known inductive construction process of 'cutting-and-stacking with a single tower' for rank-one maps. It was introduced in [dJ] (see also [Da1] and a survey [Da3]) as a convenient tool to produce rank-one actions of groups with torsions or non-Abelian groups. We show first that the restriction of T to the center of $H_3(\mathbb{R})$ is mixing. Then we adapt Ryzhikov's idea from [Ry1] to deduce that the entire action is mixing too.

Since the discrete countable Heisenberg group $H_3(\mathbb{Z})$ is a co-compact lattice in $H_3(\mathbb{Z})$[1], the restriction of a mixing zero-entropy action of $H_3(\mathbb{R})$ to $H_3(\mathbb{Z})$ yields a mixing zero-entropy action of $H_3(\mathbb{Z})$. Thus it follows from Theorem 0.1 that $H_3(\mathbb{R})$ and $H_3(\mathbb{Z})$ are both in the list of solutions of Problem 1.

In order to state the next problem considered in this paper we recall some definitions from the theory of joining of dynamical systems (see [dJRu], [Gl], [Th], [dR]). Given an ergodic action $T = (T_g)_{g \in G}$ of a locally compact second countable

[1] $H_3(\mathbb{Z})$ is defined in a similar way as $H_3(\mathbb{R})$ but with $a, b, c \in \mathbb{Z}$.

group G on a standard probability space $(X, \mathfrak{B}, \mu)$, we denote by $C(T)$ the centralizer of T, i.e. the set of μ-preserving invertible transformations of X commuting with T_g for each $g \in G$. Given a transformation $R \in C(T)$, we denote by μ_R the corresponding off-diagonal measure on $X \times X$, i.e. $\mu_R(A \times B) := \mu(RA \cap B)$. A $(T_g \times T_g)_{g \in G}$-invariant measure on $X \times X$ whose marginal on each copy of X is μ is called a *2-fold self-joining* of T. For $l > 1$, l-fold self-joinings of T are defined in a similar way. Let $J_2^e(T)$ denote the set of ergodic 2-fold self-joinings of T. If $J_2^e(T) \subset \{\mu_R \mid R \in C(R)\} \cup \{\mu \times \mu\}$ then T is called 2-fold simple. If T is 2-fold simple and for each $l > 1$, the l-fold Cartesian power of μ is the only l-fold self-joining of T whose projection on the product of any two copies of X in X^l equals $\mu \times \mu$ then T is called *simple*. If T is simple and $C(T) \subset \{T_g \mid g \in G\}$ then T is said to have the property of *minimal self-joinings* (MSJ).

In [dJ], A. del Junco raised the following problem.

Problem 3. *Given a locally compact second countable group G and a closed non-compact subgroup $L \subset G$, is there a free action T of G such that the sub-action $(T_l)_{l \in L}$ is weakly mixing and 2-fold simple and the centralizer of this sub-action is $\{T_h \mid h \in C_G(L)\}$, where $C_G(L)$ stands for the centralizer of L in G, i.e. $C_G(L) = \{g \in G \mid gl = lg \text{ for all } l \in L\}$.*

He shows that for some special pairs $L \subset G$, solutions of Problem 3 lead to non-trivial counterexamples in ergodic theory [dJ]. In the most interesting cases L is the center of G and hence $C_G(L) = G$. We now write the second main result of the paper.

Theorem 0.2. *There exists rank-one mixing action T of $H_3(\mathbb{R})$ such that the following are satisfied.*

(i) *The flow $(T_{c(t)})_{t \in \mathbb{R}}$ is simple and $C((T_{c(t)})_{t \in \mathbb{R}}) = \{T_g \mid g \in H_3(\mathbb{R})\}$.*
(ii) *The transformation $T_{c(1)}$ is simple and $C(T_{c(1)}) = \{T_g \mid g \in H_3(\mathbb{R})\}$.*
(ii) *T has MSJ.*

Thus Theorem 0.2 answers Problem 3 affirmatively in the particular case when $G = H_3(\mathbb{R})$ and L is either the center of $H_3(\mathbb{R})$ or $L = \{c(n) \mid n \in \mathbb{Z}\}$.

Next, we construct mixing rank-one infinite measure preserving actions of Heisenberg group within the framework of (C, F)-construction. We recall that an infinite measure preserving action T of $H_3(\mathbb{R})$ is called *mixing* (or 0-*type*) if $\mu(T_g A \cap B) \to 0$ as $g \to \infty$ for all subsets $A, B \subset X$ of finite measure. Mixing actions in infinite measure need not be ergodic (see [DaSi2] and references therein) however the rank one implies ergodicity. We note that while the construction of the finite measure preserving mixing actions of $H_3(\mathbb{R})$ in Theorem 0.1 is of stochastic nature (as in [Or], [Ru], [dJ]), all the parameters of the infinite measure preserving mixing actions of $H_3(\mathbb{R})$ are determined *explicitly* (effectively). For that we apply a technique of *fast-growing spacers* suggested by V. Ryzhikov in [Ry4] for the case of $\mathbb{Z}$-actions. As an application we obtain the following corollary.

Corollary 0.3. *There exist mixing Poisson and mixing Gaussian (probability preserving) actions of $H_3(\mathbb{R})$.*

We also consider a problem of *asymmetry* for ergodic actions of Heisenberg group.

Theorem 0.4. *There is a rigid weakly mixing rank-one action T of $H_3(\mathbb{R})$ such that the transformation $T_{c(1)}$ is ergodic and non-conjugate to its inverse $T_{c(1)}^{-1}$.*

To prove this theorem we 'incorporate' V. Ryzhikov's example of an asymmetric rank-one transformation from [Ry3] (see also [DaRy] for a similar example of an asymmetric rank-one flow) into the construction of mixing actions of $H(\mathbb{R})$ from Theorem 0.1.

The outline of the paper is the following. Sections 1 and 2 are preliminary. We describe the unitary dual of $H_3(\mathbb{R})$ and state the spectral decomposition theorem for unitary representations of $H_3(\mathbb{R})$ in Section 1. In Section 2 we remind the (C,F)-construction for locally compact group actions. Sections 3 and 4 are devoted to the proof of Theorems 0.1 and 0.2 respectively. In Section 5 we construct a mixing rank-one infinite measure preserving action of $H_3(\mathbb{R})$ and deduce Corollary 0.3. Section 6 is devoted to asymmetric actions of $H_3(\mathbb{R})$. We prove there Theorem 0.4. In Section 7 we apply the spectral decomposition to the Koopman representations of $H_3(\mathbb{R})$. Section 8 consists of concluding remarks and open problems.

1. Heisenberg group and its unitary dual

The Lie algebra of $H_3(\mathbb{R})$ is

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}.$$

We endow it with the natural topology. Then the exponential map $\exp : \mathfrak{h}_3 \to H_3(\mathbb{R})$ is a homeomorphism.

The subgroups $H_{2,a} := \{a(t_1)c(t_3) \mid t_1, t_3 \in \mathbb{R}\}$ and $H_{2,b} := \{b(t_2)c(t_3) \mid t_2, t_3 \in \mathbb{R}\}$ are both Abelian, normal and closed in $H_3(\mathbb{R})$. The corresponding group extensions

$$\begin{aligned} &0 \to H_{2,a} \to H_3(\mathbb{R}) \to H_3(\mathbb{R})/H_{2,a} \to 0 \quad \text{and} \\ &0 \to H_{2,b} \to H_3(\mathbb{R}) \to H_3(\mathbb{R})/H_{2,b} \to 0 \end{aligned}$$

are both split. This implies that $H_3(\mathbb{R})$ is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where the homomorphism $A : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ is given by $A(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$. The subgroups $H_{2,a}$ and $H_{2,b}$ are automorphic in $H_3(\mathbb{R})$, i.e. there is an isomorphism θ of $H_3(\mathbb{R})$ with $\theta(H_{2,a}) = H_{2,b}$. We define θ by setting $\theta(a(t)) := b(t)$, $\phi(b(t)) := a(t)$ and $\theta(c(t)) := c(-t)$ for all $t \in \mathbb{R}$. To put it in other way,

$$\theta \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & ab - c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

We call θ the *flip* in $H_3(\mathbb{R})$. We note that $\theta^2 = \mathrm{id}$.

The set of unitarily equivalent classes of irreducible (weakly continuous) representations of $H_3(\mathbb{R})$ is called the *unitary dual* of $H_3(\mathbb{R})$. It is denoted by $\widehat{H_3(\mathbb{R})}$. The irreducible unitary representations of $H_3(\mathbb{R})$ are well known. They consist of a family of 1-dimensional representations $\pi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$ and a family of infinite dimensional representations π_γ, $\gamma \in \mathbb{R} \setminus \{0\}$ as follows [Ki]:

$$\begin{aligned} \pi_{\alpha,\beta}(c(t_3)b(t_2)a(t_1)) &:= e^{i(\alpha t_1 + \beta t_2)} \quad \text{and} \\ (\pi_\gamma(c(t_3)b(t_2)a(t_1))f)(x) &:= e^{i\gamma(t_3 + t_2 x)} f(x + t_1), \quad f \in L^2(\mathbb{R}, \lambda_{\mathbb{R}}). \end{aligned}$$

Thus we can identify $\widehat{H_3(\mathbb{R})}$ with the disjoint union $\mathbb{R}^2 \sqcup \mathbb{R}^*$. There is a natural Borel σ-algebra on the unitary dual of each locally compact second countable group [Ma]. In the case of the Heisenberg group this Borel σ-algebra coincides with the standard σ-algebra of Borel subsets in $\mathbb{R}^2 \sqcup \mathbb{R}^*$. Given an arbitrary unitary representation $U = (U(g))_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ in a separable Hilbert space $\mathcal{H}$, there are a measure σ_U on $\widehat{H_3(\mathbb{R})}$ (i.e. two measures $\sigma_U^{1,2}$ on $\mathbb{R}^2$ and σ_U^3 on $\mathbb{R}^*$) and a map $l_U : \widehat{H_3(\mathbb{R})} \to \mathbb{N} \cup \{\infty\}$ (i.e. two maps $l_U^{1,2} : \mathbb{R}^2 \ni (x, y) \mapsto l_U^{1,2}(x, y) \in \mathbb{N} \cup \{\infty\}$ and $l_U^3 : \mathbb{R}^* \ni z \mapsto l_U^3(z) \in \mathbb{N} \cup \{\infty\}$) such that the following decompositions hold up to unitary equivalence

$$\mathcal{H} = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l_U^{1,2}(\alpha,\beta)} \mathbb{C}\, d\sigma_U^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l_U^3(\gamma)} L^2(\mathbb{R}, \lambda_{\mathbb{R}})\, d\sigma_U^3(\gamma) \quad \text{and}$$

$$U(g) = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l_U^{1,2}(\alpha,\beta)} \pi_{\alpha,\beta}(g)\, d\sigma_U^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l_U^3(\gamma)} \pi_\gamma(g)\, d\sigma_U^3(\gamma).$$

The equivalence class of σ_U is called the *maximal spectral type* of U. The map l_U is called the *multiplicity function* of U. The essential range if l_U is called the *set of spectral multiplicities* of U. The maximal spectral type and the multiplicity function of U (σ_U-mod 0) are both determined uniquely by the unitarily equivalent class of U.

Given a measure preserving action T of $H_3(\mathbb{R})$ on a standard probability space $(X, \mathfrak{B}, \mu)$, we can associate a unitary representation U_T of of $H_3(\mathbb{R})$ in $L^2(X, \mu)$ by setting $U_T(g)f := f \circ T_g^{-1}$. It is called the Koopman representation of $H_3(\mathbb{R})$ associated with T. The maximal spectral type of U_T is called the maximal spectral type of T and the maximal spectral type of the restriction of U_T to the subspace of zero mean functions in $L^2(X, \mu)$ is called the restricted maximal spectral type of T.

2. (C, F)-construction

Let G be a unimodular l.c.s.c. amenable group. Fix a σ-finite Haar measure λ_G on it. Given two subsets $E, F \subset G$, by EF we mean their algebraic product, i.e. $EF = \{ef \mid e \in E, f \in F\}$. The set $\{e^{-1} \mid e \in E\}$ is denoted by E^{-1}. If E is a singleton, say $E = \{e\}$, then we will write eF for EF and Fe for FE.

To define a (C, F)-action of G we need two sequences $(F_n)_{n \geq 0}$ and $(C_n)_{n>0}$ of subsets in G such that the following conditions are satisfied:

(I) $(F_n)_{n=0}^{\infty}$ is a Følner sequence in G,
(II) C_n is finite and $\#C_n > 1$,
(III) $F_n C_{n+1} \subset F_{n+1}$,
(IV) $F_n c \cap F_n c' = \emptyset$ for all $c \neq c' \in C_{n+1}$.

We put $X_n := F_n \times \prod_{k>n} C_k$, endow X_n with the standard Borel product σ-algebra and define a Borel embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, \dots). \tag{2-1}$$

It is well defined due to (III) and (IV). Then we have $X_1 \subset X_2 \subset \cdots$. Hence $X := \bigcup_n X_n$ endowed with the natural Borel σ-algebra, say $\mathfrak{B}$, is a standard Borel

space. Given a Borel subset $A \subset F_n$, we put

$$[A]_n := \{x \in X \mid x = (f_n, c_{n+1}, c_{n+2} \dots) \in X_n \text{ and } f_n \in A\}$$

and call this set an *n-cylinder*. It is clear that the σ-algebra $\mathfrak{B}$ is generated by the family of all cylinders.

Now we are going to define a 'canonical' measure on $(X, \mathfrak{B})$. Let κ_n stand for the equidistribution on C_n. Let $\nu_n := (\#C_1 \cdots \#C_n)^{-1}\lambda_G \restriction F_n$. We define an infinite product measure μ_n on X_n by setting $\mu_n := \nu_n \times \kappa_{n+1} \times \kappa_{n+2} \times \cdots$, $n \in \mathbb{N}$. Then the embeddings (2-1) are all measure preserving.

Hence a σ-finite measure μ on X is well defined by the restrictions $\mu \restriction X_n := \mu_n$, $n \in \mathbb{N}$. To put it in another way, $(X, \mu) = \operatorname{inj\,lim}_n (X_n, \mu_n)$. Since

$$\mu_{n+1}(X_{n+1}) = \frac{\nu_{n+1}(F_{n+1})}{\nu_{n+1}(F_n C_{n+1})}\mu_n(X_n) = \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}}\mu_n(X_n),$$

it follows that μ is finite if and only if

$$\text{(2-2)} \qquad \prod_{n=0}^{\infty} \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}} < \infty, \text{ i.e., } \sum_{n=0}^{\infty} \frac{\lambda_G(F_{n+1} \setminus (F_n C_{n+1}))}{\lambda_G(F_n)\#C_{n+1}} < \infty.$$

To construct a μ-preserving action of G on (X, μ), we fix a filtration $K_1 \subset K_2 \subset \cdots$ of G by compact subsets. Thus $\bigcup_{m=1}^{\infty} K_m = G$. Given $n, m \in \mathbb{N}$, we set

$$D_m^{(n)} := \left(\bigcap_{k \in K_m} (k^{-1}F_n) \cap F_n \right) \times \prod_{k>n} C_k \subset X_n \text{ and}$$

$$R_m^{(n)} := \left(\bigcap_{k \in K_m} (kF_n) \cap F_n \right) \times \prod_{k>n} C_k \subset X_n.$$

It is easy to verify that $D_{m+1}^{(n)} \subset D_m^{(n)} \subset D_m^{(n+1)}$ and $R_{m+1}^{(n)} \subset R_m^{(n)} \subset R_m^{(n+1)}$. We define a Borel mapping $K_m \times D_m^{(n)} \ni (g, x) \mapsto T_{m,g}^{(n)} x \in R_m^{(n)}$ by setting for $x = (f_n, c_{n+1}, c_{n+2}, \dots)$,

$$T_{m,g}^{(n)}(f_n, c_{n+1}, c_{n+2} \dots) := (gf_n, c_{n+1}, c_{n+2}, \dots).$$

Now let $D_m := \bigcup_{n=1}^{\infty} D_m^{(n)}$ and $R_m := \bigcup_{n=1}^{\infty} R_m^{(n)}$. Then a Borel mapping

$$T_{m,g} : K_m \times D_m \ni (g, x) \mapsto T_{m,g}x \in R_m$$

is well defined by the restrictions $T_{m,g} \restriction D_m^{(n)} := T_{m,g}^{(n)}$ for $g \in K_m$ and $n \geq 1$. It is easy to see that $D_m \supset D_{m+1}$, $R_m \supset R_{m+1}$ and $T_{m,g} \restriction D_{m+1} = T_{m+1,g}$ for all m. It follows from (I) that $\mu_n(X_n \setminus D_m^{(n)}) \to 0$ and $\mu_n(X_n \setminus R_m^{(n)}) \to 0$ as $n \to \infty$. Hence $\mu(X \setminus D_m) = \mu(X \setminus R_m) = 0$ for all $m \in \mathbb{N}$. Finally we set $\widehat{X} := \bigcap_{m=1}^{\infty} D_m \cap \bigcap_{m=1}^{\infty} R_m$ and define a Borel mapping $T : G \times \widehat{X} \ni (g, x) \to T_g x \in \widehat{X}$ by setting $T_g x := T_{m,g} x$ for some (and hence any) m such that $g \in K_m$. It is clear that $\mu(X \setminus \widehat{X}) = 0$. Thus, we obtain that $T = (T_g)_{g \in G}$ is a free Borel measure preserving action of G on a μ-conull subset of the standard σ-finite space $(X, \mathfrak{B}, \mu)$. It is easy to see that T does not depend on the choice of filtration $(K_m)_{m=1}^{\infty}$. Throughout the paper we do not distinguish between two measurable sets (or mappings) which agree almost everywhere.

Definition 2.1. T is called the *(C, F)-action of G associated with* $(C_{n+1}, F_n)_{n=0}^{\infty}$.

We now list some basic properties of (X, μ, T). Given Borel subsets $A, B \subset F_n$, we have

$$[A \cap B]_n = [A]_n \cap [B]_n, \ [A \cup B]_n = [A]_n \cup [B]_n, \tag{2-3}$$

$$[A]_n = [AC_{n+1}]_{n+1} = \bigsqcup_{c \in C_{n+1}} [Ac]_{n+1}, \tag{2-4}$$

$$T_g[A]_n = [gA]_n \text{ if } gA \subset F_n, \tag{2-5}$$

$$\mu([A]_n) = \#C_{n+1} \cdot \mu([Ac]_{n+1}) \text{ for every } c \in C_{n+1}, \tag{2-6}$$

$$\mu([A]_n) = \frac{\lambda_G(A)}{\lambda_G(F_n)} \mu(X_n). \tag{2-7}$$

In case $G = \mathbb{Z}$, it is easy to notice a similarity between the (C, F)-construction and the classical cutting-and-stacking construction of rank-one transformations. Indeed, F_{n-1} (or, more precisely, the set of $(n-1)$-cylinders) corresponds to the levels of the $(n-1)$-tower and C_n corresponds to the locations of the copies of F_{n-1} inside the n-th tower F_n. (The copies $F_{n-1}c$, $c \in C_n$, are disjoint by (IV) and they sit inside F_n by (III).) The remaining part of F_n, i.e. $F_n \setminus (F_{n-1}C_n)$, is the set of spacers in the n-th tower.

Each (C, F)-action is of rank one.

3. Mixing rank-one actions of Heisenberg group

Given three positive reals α, β and γ, we set

$$\begin{aligned} I(\alpha, \beta, \gamma) &:= \{c(t_3)b(t_2)a(t_1) \mid |t_1| \le \alpha, |t_2| \le \beta, |t_3| \le \gamma\} \\ &= \left\{ \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, |t_1| \le \alpha, |t_2| \le \beta, |t_3| \le \gamma \right\}. \end{aligned}$$

It is easy to verify that

$$I(\alpha, \beta, \gamma) I(\alpha', \beta', \gamma') \subset I(\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta') \quad \text{and}$$
$$I(\alpha, \beta, \gamma)^{-1} \subset I(\alpha, \beta, \gamma + \alpha\beta).$$

Now we define a map $\phi_{\alpha,\beta,\gamma} : \mathbb{Z}^3 \to H_3(\mathbb{R})$ by setting

$$\phi_{\alpha,\beta,\gamma}(j_1, j_2, j_3) := c(2\gamma j_3) b(2\beta j_2) a(2\alpha j_1), \quad j_1, j_2, j_3 \in \mathbb{Z}.$$

This map is not a group homomorphism. However for all $j_1, j_2, j_3, p \in \mathbb{Z}$, we have

$$\begin{aligned} \phi_{\alpha,\beta,\gamma}(0,0,p)\phi_{\alpha,\beta,\gamma}(j_1, j_2, j_3) &= \phi_{\alpha,\beta,\gamma}(0,0,p)\phi_{\alpha,\beta,\gamma}(j_1, j_2, j_3) \\ &= \phi_{\alpha,\beta,\gamma}(j_1, j_2, p + j_3). \end{aligned}$$

The following tiling property holds:

$$H_3(\mathbb{R}) = \bigsqcup_{z \in \mathbb{Z}^3} I(\alpha, \beta, \gamma)\phi_{\alpha,\beta,\gamma}(z).$$

We now choose a Haar measure λ on $H_3(\mathbb{R})$ in such a way that $\lambda(I(\alpha, \beta, \gamma)) = 8\alpha\beta\gamma$ for some (and hence for all) positive reals α, β, γ.

The proof of the following lemma is straightforward.

Lemma 3.1. *Let $\alpha_n, \beta_n, \gamma_n \to \infty$, $\alpha_n/\gamma_n \to 0$ and $\beta_n/\gamma_n \to 0$. Then $(I(\alpha_n, \beta_n, \gamma_n))_{n=1}^{\infty}$ is a two-sided Følner sequence in $H_3(\mathbb{R})$. Moreover, if $(\alpha_n\beta_n)/\gamma_n \to 0$ then*

$$\frac{\lambda(I(\alpha_n, \beta_n, \gamma_n) \triangle I(\alpha_n, \beta_n, \gamma_n)^{-1})}{\lambda(I(\alpha_n, \beta_n, \gamma_n))} \to 0.$$

We now construct inductively the sequence $(C_{n+1}, F_n)_{n=0}^{\infty}$. Suppose that on the n-th step we already defined F_0, $(C_j, F_j)_{j=1}^{n}$ and an auxiliary sequence $(\widetilde{F}_j)_{j=0}^{n-1}$ such that $F_j = I(\alpha_j, \alpha_j, \gamma_j)$ for some $\alpha_j, \gamma_j > 0$, $0 \leq j \leq n$, and $\widetilde{F}_j = I(\widetilde{\alpha}_j, \widetilde{\alpha}_j, \widetilde{\gamma}_j)$ for for some $\widetilde{\alpha}_j, \widetilde{\gamma}_j > 0$, $0 \leq j < n$. Suppose also that F_j is equipped with a finite Borel partition ξ_j, $0 \leq j < n$. Our purpose is to construct $\xi_n, \widetilde{F}_n, C_{n+1}$ and F_{n+1}.

Let ξ_n be a finite Borel partition of F_n such that the following are satisfied:

(i) the diameter (with respect to a natural metric on $h_3(\mathbb{R})$) of each atom of ξ_n is less than $\frac{1}{n}$ and

(iii) for each atom A of ξ_{n-1} and each element $c \in C_n$, the subset $Ac \subset F_n$ is ξ_n-measurable.

Next, we introduce an auxiliary set

$$S_n := I((2n+1)\widetilde{\alpha}_{n-1}, (2n+1)\widetilde{\alpha}_{n-1}, (2n+1)\widetilde{\gamma}_{n-1}).$$

Then we select $\widetilde{\alpha}_n$ and $\widetilde{\gamma}_n$ to be the smallest positive reals such that

$$F_n S_n \subset I(\widetilde{\alpha}_n, \widetilde{\alpha}_n, \widetilde{\gamma}_n).$$

Now we set $\widetilde{F}_n := I(\widetilde{\alpha}_n, \widetilde{\alpha}_n, \widetilde{\gamma}_n)$ and $\phi_n := \phi_{\widetilde{\alpha}_n, \widetilde{\alpha}_n, \widetilde{\gamma}_n}$. Next, for some integer $r_n > 0$ (to be specified below), we let

$$H_n := \{(t_1, t_2, t_3) \in \mathbb{Z}^3 \mid |t_1| < n^3, |t_2| < n^3, |t_3| < r_n\}.$$

Suppose we are given a mapping $s_n : H_n \to S_n$. Then we define another mapping $c_{n+1} : H_n \to H_3(\mathbb{R})$ by setting $c_{n+1}(h) := s_n(h)\phi_n(h)$. We now set

$$C_{n+1} := c_{n+1}(H_n).$$

Finally, let $F_{n+1} := I(\alpha_{n+1}, \alpha_{n+1}, \gamma_{n+1})$, where α_{n+1} and γ_{n+1} are the smallest positive reals such that $I(\alpha_{n+1}, \alpha_{n+1}, \gamma_{n+1}) \supset F_n C_{n+1}$. It remains to specify r_n and s_n. For that we need an auxiliary lemma from [dJ]. In order to state it we first introduce some notation. Given a finite measure ν on a finite set D, we let $\|\nu\|_1 := \sum_{d \in D} |\nu(d)|$. Given a finite set Y and a mapping $s : Y \to D$, let $\mathrm{distr}_{y \in Y} s(y)$ denote the image of the equidistribution on Y under s, i.e.

$$(\mathrm{distr}_{y \in Y} s(y))(d) := \frac{\#(s^{-1}(\{d\}))}{\#Y} \quad \text{for each} \quad d \in D.$$

Lemma 3.2 [dJ]**.** *Let D be a finite set. Then given $\epsilon > 0$ and $\delta > 0$, there is $R > 0$ such that for each $r > R$, there exists a map $s : \{-r, -r+1, \dots, r\} \to D$ such that*

$$\|\mathrm{distr}_{0\le t<N}(s(h+t), s(h'+t)) - \lambda_D \times \lambda_D\|_1 < \epsilon$$

for each $N > \delta r$ and $h \neq h'$ with $h + N < r$ and $h' + N < r$.

For a finite subset D in S_n, we denote by λ_D the corresponding normalized *Dirac comb*, i.e. a measure on S_n given by $\lambda_D(A) := \#(A \cap D)/\#D$ for each subset $A \subset S_n$. Given two subsets $A, B \subset F_n$, we let

$$f_{A,B}(x, y) := \frac{\lambda(Ax \cap By)}{\lambda(F_n)}$$

for all $x, y \in S_n$. Now we choose a finite subset D_n in S_n such that

$$\left|\int_{S_n\times S_n} f_{A,B}\, d\lambda_{D_n} d\lambda_{D_n} - \frac{1}{\lambda(S_n)^2}\int_{S_n\times S_n} f_{A,B}\, d\lambda d\lambda\right| \le \frac{1}{n} \tag{3-1}$$

for all ξ_n-measurable subsets $A, B \subset F_n$ with $AS_n \cup BS_n \subset F_n$. It follows from Lemma 3.2 that there exist $r_n > 0$ and a mapping $s_n : H_n \to D_n$ such that $s_n(t_1, t_2, t_3) = s_n(0, 0, t_3)$ for all $(t_1, t_2, t_3) \in H_n$ and

$$\|\mathrm{distr}_{0\le t<N}(s_n(h + (0, 0, t)), s_n(h' + (0, 0, t))) - \lambda_{D_n} \times \lambda_{D_n}\|_1 < \frac{1}{n} \tag{3-2}$$

for each $N > n^{-2} r_n$ and $h, h' \in H_n$ with $h_3 \neq h_3'$ and $h_3 + N < r_n$ and $h_3' + N < r_n$. Here h_3 and h_3' denote the third coordinate of h and h' respectively.

Thus a sequence $(C_{n+1}, F_n)_{n=0}^\infty$ is well defined. By Lemma 3.2 we may assume without loss of generality that $r_n \to \infty$ faster than exponentially. Then the condition (I) from Section 2 is satisfied by Lemma 3.1. It is straightforward to check that the conditions (II)–(IV) and (2-2) are also satisfied. Hence the (C, F)-action $T = (T_g)_{g\in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ associated with $(C_{n+1}, F_n)_{n=0}^\infty$ is well defined on a standard non-atomic probability space $(X, \mathfrak{B}, \mu)$.

We will need the following technical lemma.

Lemma 3.3. *Let A, B and S be subsets of finite Haar measure in $H_3(\mathbb{R})$. Then*

$$\int_{S\times S} \lambda(At_1 \cap Bt_2)\, d\lambda(t_1) d\lambda(t_2) = \int_{A\times B} \lambda(aS \cap bS)\, d\lambda(a) d\lambda(b)$$

Proof. Without loss of generality we may assume that the subsets A, B and S are open and relatively compact in $H_3(\mathbb{R})$. Let

$$\Omega_1 := \{(t_1, t_2, t_3) \in H_3(\mathbb{R})^3 \mid t_1, t_2 \in S, t_3 \in At_1 \cap Bt_2\},$$
$$\Omega_2 := \{(a, b, c) \in H_3(\mathbb{R})^3 \mid a \in A, b \in B, c \in aS \cap bS\}.$$

Then Ω_j is open and relatively compact subset in $H_3(\mathbb{R})^3$, $j = 1, 2$. Choosing an appropriate normalization of λ we may assume without loss of generality that $\lambda^3(\Omega_1 \cup \Omega_2) = 1$. For each $\epsilon > 0$, we find a lattice Γ in $H_3(\mathbb{R})$ such that

$$\left|\frac{\#(\Gamma^3 \cap \Omega_j)}{\#(\Gamma^3 \cap (\Omega_1 \cup \Omega_2))} - \lambda^3(\Omega_j)\right| < \epsilon, \quad j = 1, 2.$$

Define a measure ν on $H_3(\mathbb{R})$ by setting $\nu(A) := \alpha\#(A \cap \Gamma)$ for all Borel subsets $A \subset H_3(\mathbb{R})$, where the normalizing constant $\alpha > 0$ is chosen in such a way that $\nu^3(\Omega_1 \cup \Omega_2) = 1$. Then

$$|\nu^3(\Omega_j) - \lambda^3(\Omega_j)| < \epsilon, \quad j = 1, 2. \tag{3-3}$$

We set $A_\Gamma := A \cap \Gamma$, $B_\Gamma := B \cap \Gamma$ and $S_\Gamma := S \cap \Gamma$. Then

$$\begin{aligned} &\nu(At_1 \cap Bt_2) = \nu(A_\Gamma t_1 \cap B_\Gamma t_2) \quad \text{and} \\ &\nu(t_1 A \cap t_2 B) = \nu(t_1 A_\Gamma \cap t_2 B_\Gamma) \end{aligned} \tag{3-4}$$

whenever $t_1, t_2 \in \Gamma$. Applying Fubini theorem, (3-3) and (3-4) we obtain that

$$\begin{aligned} \int_{S\times S} \lambda(At_1 \cap Bt_2)\, d\lambda(t_1) d\lambda(t_2) &= \lambda^3(\Omega_1) \\ &= \nu^3(\Omega_1) \pm \epsilon \\ &= \int_{S_\Gamma \times S_\Gamma} \nu(At_1 \cap Bt_2)\, d\nu(t_1) d\nu(t_2) \pm \epsilon \\ &= \int_{S_\Gamma \times S_\Gamma} \nu(A_\Gamma t_1 \cap B_\Gamma t_2)\, d\nu(t_1) d\nu(t_2) \pm \epsilon \\ &= \sum_{a \in A_\Gamma} \sum_{b \in B_\Gamma} \int_{S_\Gamma \times S_\Gamma} \nu(\{at_1\} \cap \{bt_2\})\, d\nu(t_1) d\nu(t_2) \pm \epsilon \\ &= \int_{A_\Gamma \times B_\Gamma} \nu(aS_\Gamma \cap bS_\Gamma)\, d\nu(a) d\nu(b) \pm \epsilon \\ &= \int_{A \times B} \nu(aS \cap bS)\, d\nu(a) d\nu(b) \pm \epsilon \\ &= \nu^3(\Omega_2) \pm \epsilon \\ &= \lambda^3(\Omega_2) \pm 2\epsilon \\ &= \int_{A \times B} \lambda(aS \cap bS)\, d\lambda(a) d\lambda(b) \pm 2\epsilon. \end{aligned}$$

□

We need some notation. Given subsets $A_n, A'_n \subset F_n$, we write $A_n \sim A'_n$ as $n \to \infty$ if $\lim_{n\to\infty} \lambda(A_n \triangle A'_n)/\lambda(F_n) = 0$. This property implies that $\mu([A_n]_n \triangle [A'_n]_n) \to 0$ as $n \to \infty$. Let $e_3 := (0, 0, 1) \in \mathbb{Z}^3$.

Lemma 3.4.

$$\sup_{A^*, B^* \subset F_{n-1}} |\mu(T_{\phi_n(e_3)}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})| \to 0$$

as $n \to \infty$. More generally, given a sequence of subsets $H_n^* \subset H_n$ such that

$$\frac{\#(H_n^* \cap (H_n^* - e_3))}{\#H_n^*} \to 1 \quad \text{and} \quad \frac{\#H_n^*}{\#H_n} \to \delta$$

for some $\delta > 0$, we let $C_n^* := \phi_n(H_n^*)$. Then

$$\sup_{A^*, B^* \subset F_{n-1}} |\mu(T_{\phi_n(e_3)}[A^* C_n^*]_n \cap [B^*]_{n-1}) - \mu([A^* C_n^*]_n)\mu([B^*]_{n-1})| \to 0$$

as $n \to \infty$.

Proof. Fix $n > 0$ and two Borel subsets $A, B \subset F_n$. Since $\phi_n(e_3)$ belongs to the center of $H_3(\mathbb{R})$,

$$\phi_n(e_3)c_{n+1}(h) = s_n(h)s_n(h+e_3)^{-1}c_{n+1}(h+e_3) \tag{3-5}$$

for all $h \in \mathbb{Z}^3$ such that $h, h+e_3 \in H_n$. We let

$$F_n^\circ := \{f \in F_n \mid fS_nS_n^{-1} \subset F_n\}.$$

In other words, F_n° in an *interior* of F_n. Of course, $F_n \sim F_n^\circ$ as $n \to \infty$. We now put $A^\circ := A \cap F_n^\circ$, $B^\circ := B \cap F_n^\circ$ and $H_n(e_3) := H_n \cap (H_n - e_3)$. Applying the standard approximation argument we may assume without loss of generality that A° and B° are ξ_n-measurable. It follows from (2-4) and (2-5) that

$$\begin{aligned}\mu(T_{\phi_n(e_3)}[A]_n \cap [B]_n) &= \sum_{h \in H_n} \mu(T_{\phi_n(e_3)}[Ac_{n+1}(h)]_{n+1} \cap [B]_n)\\ &= \sum_{h \in H_n(e_3)} \mu([\phi_n(e_3)A^\circ c_{n+1}(h)]_{n+1} \cap [B]_n) + \overline{o}(1).\end{aligned} \tag{3-6}$$

By $\overline{o}(1)$ here and below we mean a sequence that tends to 0 uniformly in $A \subset F_n$ and $B \subset F_n$. We now deduce from (3-5), (3-6) and (2-3), (2-4), (2-6) that

$$\mu(T_{\phi_n(e_3)}[A]_n \cap [B]_n) = \frac{1}{\#H_n} \sum_{h \in H_n(e_3)} \mu([(A^\circ s_n(h)s_n(h+e_3)^{-1} \cap B)]_n) + \overline{o}(1).$$

Now (3-2), (2-7) and (3-1) yield

$$\begin{aligned}\mu(T_{\phi_n(e_3)}[A]_n \cap [B]_n) &= \frac{1}{(\#D_n)^2} \sum_{d_1, d_2 \in D_n} \mu([(A^\circ d_1 d_2^{-1} \cap B)]_n) + \overline{o}(1)\\ &= \frac{1}{(\#D_n)^2} \sum_{d_1, d_2 \in D_n} \frac{\lambda(A^\circ d_1 \cap B^\circ d_2)}{\lambda(F_n)} + \overline{o}(1)\\ &= \int_{S_n \times S_n} f_{A^\circ, B^\circ}\, d\lambda_{D_n}\lambda_{D_n} + \overline{o}(1)\\ &= \frac{1}{\lambda(S_n)^2} \int_{S_n \times S_n} f_{A^\circ, B^\circ}\, d\lambda d\lambda + \overline{o}(1)\\ &= \frac{1}{\lambda(S_n)^2} \int_{S_n \times S_n} f_{A, B}\, d\lambda d\lambda + \overline{o}(1).\end{aligned} \tag{3-7}$$

Suppose now that $A = A^*C_n$ and $B = B^*C_n$ for some subsets A^* and B^* in F_{n-1}. We say that elements c and c' of C_n are *partners* if $F_{n-1}cS_n \cap F_{n-1}c'S_n \neq \emptyset$. We then write $c \bowtie c'$. It follows that

$$\int_{S_n \times S_n} f_{A,B}\, d\lambda d\lambda = \int_{S_n \times S_n} \frac{\sum_{c \bowtie c' \in C_n} \lambda(A^*ct_1 \cap B^*c't_2)}{\lambda(F_n)}\, d\lambda(t_1)d\lambda(t_2).$$

Applying Lemma 3.3 we now obtain that

$$\text{(3-8)} \qquad \int_{S_n\times S_n} f_{A,B}\, d\lambda d\lambda = \sum_{c\bowtie c'\in C_n} \int_{A^*\times B^*} \frac{\lambda(acS_n\cap bc'S_n)}{\lambda(F_n)}\, d\lambda(a)d\lambda(b).$$

Next, we note that

$$\sup_{c,c'\in C_n}\sup_{a,b\in F_{n-1}} \left|\frac{\lambda(acS_n\cap bc'S_n)-\lambda(cS_n\cap c'S_n)}{\lambda(S_n)}\right| \to 0 \quad \text{as } n\to\infty.$$

Hence it follows from (3-7) and (3-8) that

$$\begin{aligned}&\mu(T_{\phi_n(e_3)}[A^*]_{n-1}\cap[B^*]_{n-1})\\ &\qquad = \frac{\sum_{c\bowtie c'\in C_n}}{\lambda(S_n)^2}\int_{A^*\times B^*}\frac{\lambda(cS_n\cap c'S_n)+\lambda(S_n)\cdot\overline{o}(1)}{\lambda(F_n)}\, d\lambda(a)d\lambda(b)+\overline{o}(1).\end{aligned}$$

A routine verification shows that every $c\in C_n$ has no more than $2^{10}n^3$ partners. Therefore

$$\begin{aligned}&\mu(T_{\phi_n(e_3)}[A^*]_{n-1}\cap[B^*]_{n-1})\\ &\qquad = \lambda(A^*)\lambda(B^*)\theta_n \pm \frac{2^{10}n^3\#C_n}{\lambda(S_n)^2}\cdot\frac{\lambda(F_{n-1})^2\lambda(S_n)\cdot\overline{o}(1)}{\lambda(F_n)}+\overline{o}(1)\end{aligned}$$

for some $\theta_n>0$. Since

$$\lim_{n\to\infty}\frac{n^3\lambda(F_{n-1})}{\lambda(S_n)}>0 \quad\text{and}\quad \lim_{n\to\infty}\frac{\lambda(F_{n-1})\#C_n}{\lambda(F_n)}=1,$$

we obtain

$$\mu(T_{\phi_n(e_3)}[A^*]_{n-1}\cap[B^*]_{n-1}) = \frac{\lambda(A^*)\lambda(B^*)}{\lambda(F_{n-1})^2}\theta_n'+\overline{o}(1)$$

for some $\theta_n'>0$. Substituting $A^*=B^*=F_{n-1}$ we obtain that $\theta_n\to 1$ as $n\to\infty$. Thus

$$\mu(T_{\phi_n(e_3)}[A^*]_{n-1}\cap[B^*]_{n-1}) = \mu([A^*]_{n-1})\mu([B^*]_{n-1})+\overline{o}(1),$$

and the first claim of the lemma is proved. The second claim is proved in a similar way. □

Corollary 3.5. *The sequence $(\phi_n(e_3))_{n=1}^\infty$ is mixing for T, i.e. $\mu(T_{\phi_n(e_3)}A\cap B)\to\mu(A)\mu(B)$ as $n\to\infty$ for all Borel subsets $A,B\subset X$. Hence the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is weakly mixing.*

We now refine this corollary.

Proposition 3.6. *The flow $(T_{c(t)})_{t\in\mathbb{R}}$ is mixing.*

Proof. Take a sequence of reals $t_n>0$ such that $g_n:=c(t_n)\in F_{n+1}\setminus F_n$, $n\in\mathbb{N}$. It suffices to show that the sequence $(g_n)_{n=1}^\infty$ (or, at least a subsequence of it) is mixing for T. We write $g_n=f_n\phi_n(0,0,j_n)$ for some $f_n\in F_n\cap\{c(t)\mid t>0\}$ and $0<j_n<r_n$. Let $h_n:=(0,0,j_n)\in\mathbb{Z}^3$, $H_n(h_n):=H_n\cap(H_n-h_n)$ and

$F_n(f_n) := F_n \cap (f_n^{-1}F_n)$. Passing to a subsequence, if necessary, we may assume without loss of generality that there exist $\delta_1 \geq 0$ and $\delta_2 \geq 0$ such that

$$\frac{\#H_n(h_n)}{\#H_n} \to \delta_1 \quad \text{and} \quad \frac{\lambda(F_n(f_n))}{\lambda(F_n)} \to \delta_2.$$

Partition C_{n+1} into three subsets C_{n+1}^1, C_{n+1}^2 and C_{n+1}^3 as follows

$$\begin{aligned}
C_{n+1}^1 &:= \{c \in C_{n+1} \mid g_n F_n c \subset F_{n+1}\phi_{n+1}(e_3)\},\\
C_{n+1}^2 &:= \{c \in C_{n+1} \mid g_n F_n c \subset F_{n+1}\},\\
C_{n+1}^3 &:= C_{n+1} \setminus (C_{n+1}^1 \cup C_{n+1}^2).
\end{aligned}$$

We will show mixing separately on each of the subsets $[F_nC_{n+1}^1]_{n+1}$, $[F_nC_{n+1}^2]_{n+1}$ and $[F_nC_{n+1}^3]_{n+1}$ of X.

We first note that $\#C_{n+1}^3/\#C_{n+1} \to 0$ as $n \to \infty$. Then (2-6) yields that

$$\lim_{n\to\infty} \sup_{A\subset F_n} \mu([AC_{n+1}^3]_{n+1}) = 0. \tag{3-9}$$

Next, we note that $\phi_{n+1}(e_3)^{-1}g_nF_nC_{n+1}^1 \subset F_{n+1}$ and hence

$$T_{g_n}[AC_{n+1}^1]_{n+1} = T_{\phi_{n+1}(e_3)}[\phi_{n+1}(e_3)^{-1}g_nAC_{n+1}^1]_{n+1}.$$

Hence, by the second claim of Lemma 3.4,

$$\lim_{n\to\infty} \sup_{A,B\subset F_n} |\mu(T_{g_n}[AC_{n+1}^1]_{n+1} \cap [B]_n) - \mu([AC_{n+1}^1]_{n+1})\mu([B]_n)| = 0. \tag{3-10}$$

It remains to consider the third case involving C_{n+1}^2. It is a routine to verify that

$$\frac{\#(C_{n+1}^2 \triangle \{c_{n+1}(h) \mid h \in H_n(h_n)\}}{\#C_{n+1}^2} \to 1,$$

If $\delta_1 = 0$ then

$$\lim_{n\to\infty} \sup_{A\subset F_n} \mu([AC_{n+1}^2]_{n+1}) = 0. \tag{3-11}$$

Consider now the case where $\delta_1 > 0$. Take two subsets $A, B \subset F_n$. Partition A into two subsets A_1 and A_2 such that $f_nA_1 \subset F_n$ and $f_nA_2 \subset F_n\phi_n(e_3)$. We note that $A_1 = A \cap F_n(f_n)$. Define F_n° in the same way as in the proof of Lemma 3.4 and set $A_1^\circ := A_1 \cap F_n^\circ$ and $B^\circ := B \cap F_n^\circ$. Slightly modifying the reasoning in the proof of Lemma 3.4 we obtain

$$\begin{aligned}
\mu(T_{g_n}[A_1C_{n+1}^2]_{n+1} \cap [B]_n) &= \sum_{h\in H_n(h_n)} \mu(T_{\phi_n(h_n)}[f_nA_1c_{n+1}(h)]_{n+1} \cap [B]_n) + \overline{o}(1)\\
&= \sum_{h\in H_n(h_n)} \mu([\phi_n(t_n)f_nA_1^\circ c_{n+1}(h)]_{n+1} \cap [B]_n) + \overline{o}(1)\\
&= \frac{\delta_1}{\#H_n(t_n)} \sum_{h\in H_n(t_n)} \mu([(f_nA_1^\circ s_n(h)s_n(h+t_n)^{-1} \cap B^\circ)]_n) + \overline{o}(1)\\
&= \frac{\delta_1}{\lambda(S_n)^2} \int_{S_n\times S_n} f_{A_1,B}\, d\lambda d\lambda + \overline{o}(1).
\end{aligned}$$

If $\delta_2 = 0$ then

$$\lim_{n\to\infty} \sup_{A\subset F_n} \mu([A_1 C^2_{n+1}]_{n+1}) = 0. \tag{3-12}$$

Consider now the case where $\delta_2 > 0$. As in the proof of Lemma 3.4 we now take $A = A^* C_n$ and $B = B^* C_n$ for some Borel subsets $A^*, B^* \subset F_{n-1}$. Let $C'_n := C_n \cap F_n(f_n)$. It follows that

$$\frac{\# C'_n}{\# C_n} \to \delta_2 \quad \text{and} \quad \sup_{A^*\subset F_{n-1}} \mu([A_1]_n \triangle [A^* C'_n]_n) \to 0$$

as $n \to \infty$. Hence $\sup_{A^*\subset F_{n-1}} |\mu([A_1]_n) - \delta_2 \mu([A^*]_{n-1})| \to 0$. Arguing as in the proof of Lemma 3.4 we obtain that

$$\sup_{A^*,B^*\subset F_{n-1}} |\mu(T_{g_n}[A^* C'_n C^2_{n+1}]_{n+1} \cap [B^*]_{n-1}) - \delta_2 \mu([A^*]_{n-1})\mu([B^*]_{n-1})| \to 0.$$

Therefore

$$\lim_{n\to\infty} \sup_{A^*,B^*\subset F_{n-1}} |\mu(T_{g_n}[A_1]_n \cap [B^*]_{n-1}) - \mu([A_1]_n)\mu([B^*]_{n-1})| = 0. \tag{3-13}$$

Since $T_{g_n}[A_2]_n = T_{\phi_n(h_n+e_3)}[\phi_n(e_3)^{-1} f_n A_2]_n$ with $\phi_n(e_3)^{-1} f_n A_2 \subset F_n$, a similar reasoning yields

$$\lim_{n\to\infty} \sup_{A^*,B^*\subset F_{n-1}} |\mu(T_{g_n}[A_2]_n \cap [B^*]_{n-1}) - \mu([A_2]_n)\mu([B^*]_{n-1})| = 0. \tag{3-14}$$

Since

$$[A^*]_{n-1} = [A^* C_n C^1_{n+1}]_{n+1} \sqcup [A^* C_n C^3_{n+1}]_{n+1} \sqcup [A_1 C^2_{n+1}]_{n+1} \sqcup [A_2 C^2_{n+1}]_{n+1},$$

it follows from (3-9)–(3-14) that

$$\lim_{n\to\infty} \sup_{A^*,B^*\subset F_{n-1}} |\mu(T_{g_n}[A^*]_{n-1} \cap [B^*]_{n-1}) - \mu([A^*]_{n-1})\mu([B^*]_{n-1})| = 0,$$

i.e. $(g_n)_{n=1}^\infty$ is a mixing sequence for T, as desired. □

We now state and prove the main result of this section. By an advice of V. Ryzhikov we deduce it from Proposition 3.6 by adapting the argument used in [Ry1, Theorem 6] and [Ry2, Theorem 4.4] to show mixing of $(T_{a(t)})_{t\in\mathbb{R}}$.

Theorem 3.7. *Let T be an action of $H_3(\mathbb{R})$ such that the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is ergodic. Then T is mixing.*

Proof. If T were not mixing then there exist $\epsilon > 0$, a sequence $g_n \to \infty$ in $H_3(\mathbb{R})$ and two subsets $A_0, B_0 \subset X$ such that

$$|\mu(A_0 \cap T_{g_n} B_0) - \mu(A_0)\mu(B_0)| > \epsilon \qquad \text{for all } n. \tag{3-15}$$

We write g_n as $g_n = a(t_{n,1})b(t_{n,2})c(t_{n,3})$ with $t_{n,1}, t_{n,2}, t_{n,3} \in \mathbb{R}$. Since the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is mixing by Proposition 3.6, we may assume without loss of generality

(passing to a subsequence, if necessary) that either $t_{n,1} \to \infty$ or $t_{n,2} \to \infty$. We consider the former case (the latter case is analogous). Select $\gamma > 0$ such that

$$\sup_{n>0} |\mu(A_0 \cap T_{g_n} B_0) - \mu(T_{b(\beta)} A_0 \cap T_{g_n} T_{b(\beta)} B_0)| < \epsilon/2 \qquad \text{for all } \beta < \gamma.$$

Consider a probability measure $\kappa_{\gamma,n}$ on $X \times X$ by setting

$$\kappa_{\gamma,n}(A \times B) := \frac{1}{\gamma} \int_0^\gamma \mu(A \cap T_{b(\beta)^{-1} g_n b(\beta)} B) \, d\beta \qquad \text{for all } A, B \subset X.$$

It is easy to see that $\kappa_{\gamma,n}$ is a 2-fold self-joining of $(T_{c(t)})_{t\in\mathbb{R}}$ and

$$|\kappa_{\gamma,n}(A_0 \times B_0) - \mu(A_0 \cap T_{g_n} B_0)| < \epsilon/2 \qquad \text{for all } n. \tag{3-16}$$

Passing to a further subsequence we may assume without loss of generality that the sequence $(\kappa_{\gamma,n})_{n\in\mathbb{N}}$ converges weakly to a 2-fold self-joining κ_γ of $(T_{c(t)})_{t\in\mathbb{R}}$. Since

$$b(\beta)^{-1} g_n b(\beta) = g_n c(\beta t_{n,1}) \qquad \text{for all } \beta \in \mathbb{R},$$

it follows that

$$\begin{aligned} \kappa_{\gamma,n}(A \times T_{c(t)} B) &= \frac{1}{\gamma} \int_0^\gamma \mu(A \cap T_{g_n c(\beta t_{n,1} + t)} B) \, d\beta \\ &= \frac{1}{\gamma} \int_{t/t_{n,1}}^{\gamma + t/t_{n,1}} \mu(A \cap T_{g_n c(\beta t_{n,1})} B) \, d\beta. \end{aligned}$$

Hence $\kappa_{\gamma,n}(A \times T_{c(t)} B) \to \kappa_\gamma(A \times B)$ as $n \to \infty$. This means that $\kappa_\gamma \circ (\mathrm{Id} \times T_{c(t)}) = \kappa_\gamma$ for all $t \in \mathbb{R}$. This yields that $\kappa_\gamma = \mu \times \mu$. We obtain a contradiction with (3-15) plus (3-16). $\square$

We recall that the group $\mathrm{Aut}(X,\mu)$ of all invertible μ-preserving transformations of X is Polish with respect to the weak topology. The *weak topology* is the weakest topology in which all the maps $\mathrm{Aut}(X,\mu) \ni S \mapsto \mu(SA \cap B) \in \mathbb{R}$, $A, B \subset X$, are continuous.

As a byproduct of the proof of Theorem 3.7, we obtain the following results.

Corollary 3.8. *Let T be an action of $H_3(\mathbb{R})$ such that the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is ergodic. Then the following are satisfied.*

(i) *Every sequence $(g_n)_{n=1}^\infty$ in $H_3(\mathbb{R})$ such that $g_n = a(t_{n,1}) b(t_{n,2}) c(t_{n,3})$ with $(t_{1,n})_{n=1}^\infty$ or $(t_{n,2})_{n=1}^\infty$ unbounded is mixing.*

(ii) *The flows $(T_{a(t)})_{t\in\mathbb{R}}$ and $(T_{b(t)})_{t\in\mathbb{R}}$ are both mixing* [Ry1, Theorem 6] *and* [Ry2, Theorem 4.4]*).*

(iii) *The weak closure of $\{T_g \mid g \in H_3(\mathbb{R})\}$ in* $\mathrm{Aut}(X,\mu)$ *is the union of $\{T_g \mid g \in H_3(\mathbb{R})\}$ and the weak closure of $\{T_{c(t)} \mid t \in \mathbb{R}\}$.*

(iv) *If T is rigid then $\{T_{c(t)} \mid t \in \mathbb{R}\}$ is rigid.*

Remark 3.9. In a similar way, we can show the following. Let T be an action of $H_3(\mathbb{R})$.

(i) If the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is mixing of order k then T is mixing of order k.

(ii) If $(T_{c(t)})_{t\in\mathbb{R}}$ is weakly mixing then the flows $(T_{a(t)})_{t\in\mathbb{R}}$ and $(T_{b(t)})_{t\in\mathbb{R}}$ are both mixing of order k [Ry1, Theorem 6] and [Ry2, Theorem 4.4]).

Then a natural question arises: how can we construct T such that $(T_{c(t)})_{t\in\mathbb{R}}$ is mixing of order k for $k>1$? For that we need to modify slightly the construction of T from Proposition 3.6. Indeed, the only modification concerns the right choice of the 'spacer mappings' s_n. Namely, we need to replace (3-2) with the following subtler condition:

$$\|\mathrm{distr}_{0\le t<N}(s_n(h^{(1)}+(0,0,t)),\dots,s_n(h^{(k)}+(0,0,t)))-\lambda^k_{D_n}\|_1<\frac{1}{n}$$

for each $N>n^{-2}r_n$ and $h^{(1)},\dots,h^{(k)}\in H_n$ with $h_3^{(1)}\ne h_3^{(2)}\ne\dots\ne h_3^{(k)}$ and $h_3^{(1)}+N<r_n,\ \dots,\ h_3^{(k)}+N<r_n$. We leave details to the readers.

We conclude this section with a simple but useful observation.

Proposition 3.10. *Let T be an action of $H_3(\mathbb{R})$. If $(T_{c(t)})_{t\in\mathbb{R}}$ is ergodic then it is weakly mixing.*

Proof. Indeed, let $f\circ T_{c(t)}=e^{ist}f$ for some $0\ne s\in\mathbb{R}$ and $0\ne f\in L^2(X,\mu)$. Then for each $g\in H_3(\mathbb{R})$, $(f\circ T_g)\circ T_{c(t)}=e^{ist}f\circ T_g$. Since the center is ergodic, it follows that $f\circ T_g=\xi(g)f$ for some $\xi(g)\in\mathbb{C}$. It is straightforward that ξ is a continuous character of $H_3(\mathbb{R})$. Since every continuous character of $H_3(\mathbb{R})$ is trivial on the center, it follows that f is invariant under $(T_{c(t)})_{t\in\mathbb{R}}$. Hence it is constant. □

4. The 'joining' structure of the action

By a polyhedron in $\mathbb{R}^3$ we mean a union of finitely many mutually disjoint convex polyhedrons. We say that a polyhedron is *rational* if the coordinates of its vertices are all rational. If no one face of a polyhedron is parallel to the line $\{0\}\times\{0\}\times\mathbb{R}\subset\mathbb{R}^3$ then we call this polyhedron *normal*. The intersection of two normal polyhedrons and the union of two disjoint normal polyhedrons is a normal polyhedron. It is a routine to verify that if A is a normal polyhedron and $g\in H_3(\mathbb{R})$ then gA and Ag are normal polyhedrons. Given a normal polyhedron A and $\epsilon>0$, there is a rational polyhedron $B\subset A$ such that

$$\sup_{t_1,t_2}\lambda_{\mathbb{R}}(\{t\mid(t_1,t_2,t)\in A\setminus B\})<\epsilon.$$

From now on all the polyhedrons are assumed to be open subsets of $\mathbb{R}^3$.

Theorem 4.1. *Let T be the action of $H_3(\mathbb{R})$ constructed in Section 3. Then the following are satisfied.*

(i) *The flow $(T_{c(t)})_{t\in\mathbb{R}}$ is simple and $C((T_{c(t)})_{t\in\mathbb{R}})=\{T_g\mid g\in H_3(\mathbb{R})\}$.*
(ii) *The transformation $T_{c(1)}$ is simple and $C(T_{c(1)})=\{T_g\mid g\in H_3(\mathbb{R})\}$.*

Proof. (i) Take an ergodic 2-fold self-joining ν for $(T_{c(t)})_{t\in\mathbb{R}}$. Identifying $H_3(\mathbb{R})$ with $\mathbb{R}^3$ via the mapping $c(t_3)b(t_2)a(t_1)\mapsto(t_1,t_2,t_3)$ we can say about polyhedrons in $H_3(\mathbb{R})$. Recall that $F_n=I(\alpha_n,\alpha_n,\gamma_n)$. We call a point $(x,x')\in X\times X$ *generic* for $(\nu,(T_{c(t)}\times T_{c(t)})_{t\in\mathbb{R}})$ if

$$\frac{n^2}{\gamma_n}\int_0^{\gamma_n/n^2}1_{[A]_m\times[A']_m}(T_{c(t)}x,T_{c(t)}x')\,d\lambda_{\mathbb{R}}(t)\to\nu([A]_m\times[A']_m)\tag{4-1}$$

for each pair of rational polyhedrons $A,A'\subset F_m$, $m\in\mathbb{N}$. It follows from the pointwise ergodic theorem for ergodic flows that ν-almost every point of $X\times X$

is generic for $(\nu, (T_{c(t)} \times T_{c(t)})_{t\in\mathbb{R}})$. Fix such a point (x, x'). We now claim that (4-1) holds for each pair of normal convex (not necessarily rational) polyhedrons $A, A' \subset F_m$. For that approximate A and A' with nested sequences of rational convex polyhedrons $A_1 \subset A_2 \subset \cdots \subset A$ and $A'_1 \subset A'_2 \subset \cdots \subset A'$ such that

$$\sup_{t_1,t_2} \lambda_{\mathbb{R}}(\{t \mid (t_1, t_2, t) \in (A \setminus A_j) \cup (A' \setminus A'_j)\}) < \frac{1}{j} \quad \text{for all } j$$

and pass to the limit in (4-1).

Since ν is a 2-fold self-joining of T, it follows that x and x' are generic points for $(\mu, T_{c(t)})_{t\in\mathbb{R}})$, i.e.

$$\frac{n^2}{\gamma_n} \int_0^{\gamma_n/n^2} 1_{[A]_m}(T_{c(t)}x)\, d\mu_{\mathbb{R}}(t) \to \mu([A]_m)$$

for each normal polyhedron A in F_m, $m \in \mathbb{N}$. If n is sufficiently large then we can write x and x' as infinite sequences $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ and $x' = (f'_n, c'_{n+1}, c'_{n+2}, \dots)$ with $f_n, f'_n \in F_n$ and $c_j, c'_j \in C_j$ for all $j > n$. Represent f_n, f'_n, c_j and c'_j as products

$$f_n = c(\tau_{3,n})b(\tau_{2,n})a(\tau_{1,n}), \quad f'_n = c(\tau'_{3,n})b(\tau'_{2,n})a(\tau'_{1,n}),$$
$$c_j = c(t_{3,j})b(t_{2,j})a(t_{1,j}), \quad c'_j = c(t'_{3,j})b(t'_{2,j})a(t'_{1,j}).$$

Since $F_n = I(\alpha_n, \alpha_n, \gamma_n)$, we obtain that $|\tau_{3,n}| < \gamma_n$ and $|\tau'_{3,n}| < \gamma_n$. Moreover, by a standard application of Borel-Cantelli lemma, we may assume without loss of generality that $\frac{\max(t_{3,j}, t'_{3,j})}{\gamma_j} < 1 - \frac{4}{j^2}$ for all $j > n$ if n is sufficiently large. Increasing n by 1 if necessary, we will assume also that $\frac{\max(\tau_{3,n}, \tau'_{3,n})}{\gamma_n} < 1 - \frac{1}{n^2}$. We set $t_n := f'_n f_n^{-1}$.

Let B and B' be two normal polyhedrons in F_m. It follows from (4-1) that for each $\epsilon > 0$ there is $n > m$ such that

$$\text{(4-2)} \qquad \frac{n^2}{\gamma_n} \int_0^{\gamma_n/n^2} 1_{[B]_m \times [B']_m}(T_{c(t)}x, T_{c(t)}x')\, d\lambda_{\mathbb{R}}(t) = \nu([B]_m \times [B']_m) \pm \epsilon.$$

Let $\widetilde{B} := BC_{m+1} \cdots C_n$ and $\widetilde{B}' := B'C_{m+1} \cdots C_n$. Then $\widetilde{B}$ and $\widetilde{B}'$ are normal polyhedrons in F_n and $[B]_m = [\widetilde{B}]_n$ and $[B']_m = [\widetilde{B}']_n$. If $g \in H_3(\mathbb{R})$ and $gf'_n \in F_n$ then $T_g x' \in [\widetilde{B}']_n$ if and only if $T_g T_{t_n} x \in [\widetilde{B}']_n$. Hence

$$\text{(4-3)} \qquad 1_{[\widetilde{B}]_n \times [\widetilde{B}']_n}(T_g x, T_g x') = 1_{T_g^{-1}[\widetilde{B}]_n \cap T_{t_n}^{-1} T_g^{-1} [\widetilde{B}']_n}(x).$$

Since n is large, $c(\gamma_n/n^2) f'_n \in F_n$ and therefore (4-2) and (4-3) yield

$$\text{(4-4)} \qquad \frac{n^2}{\gamma_n} \int_0^{\gamma_n/n^2} 1_{T_{c(t)}^{-1}([\widetilde{B}]_n \cap T_{t_n}^{-1}[\widetilde{B}']_n)}(x)\, d\lambda_{\mathbb{R}}(t) = \nu([B]_m \times [B']_m) \pm \epsilon.$$

We note that for each $g \in H_3(\mathbb{R})$ there is $M > n$ such that $gF_nC_{n+1} \cdots C_M \subset F_M$. Therefore choosing a sufficiently large M we can write for each $0 < t < \gamma_n/n^2$ that

$$T_{c(t)}^{-1}([\widetilde{B}]_n \cap T_{t_n}^{-1}[\widetilde{B}']_n) = [D]_M,$$

where $D = c(t)^{-1}(\widetilde{B}C_{n+1}\cdots C_M \cap t_n^{-1}\widetilde{B}'C_{n+1}\cdots C_M)$. Since D is a normal polyhedron in F_M and x is generic for $(\mu, T_{c(t)})_{t\in\mathbb{R}})$, we deduce from (4-4) that

$$\frac{n^2}{\gamma_n}\int_0^{\gamma_n/n^2} \mu(T_{c(t)}^{-1}([\widetilde{B}]_n \cap T_{t_n}^{-1}[\widetilde{B}']_n))\, d\lambda_{\mathbb{R}}(t) = \nu([B]_m \times [B']_m) \pm \epsilon.$$

Thus

$$\mu([B]_m \cap T_{t_n}^{-1}[B']_m) = \nu([B]_m \times [B']_m) \pm \epsilon.$$

We note that the sequence $(t_n)_{n=1}^{\infty}$ either converges to some $g \in H_3(\mathbb{R})$ (in the case when $c_j = c'_j$ for all sufficiently large j) or tends to infinity (in the case when $c_j \neq c'_j$ for infinitely many j). In the first case $\nu = \mu_{T_g}$ and in the second case $\nu = \mu \times \mu$. In the second case we use the fact that T is mixing. Thus the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is 2-fold simple and its centralizer is $\{T_g \mid g \in H_3(\mathbb{R})\}$. By [Ry4], each 2-fold simple flow is simple.

(ii) follows from (i) and [dJR, Theorem 6.1]. □

Remark 4.2. In a similar way one can show a more general fact: each mixing (C, F)-action of $H_3(\mathbb{R})$ with $F_n = I(\alpha_n, \beta_n, \gamma_n)$ and $\gamma_n \gg \alpha_n\beta_n$ is 2-fold simple.

Theorem 4.3. *Given an action T of $H_3(\mathbb{R})$, let the flow $(T_{c(t)})_{t\in\mathbb{R}}$ be simple and $C((T_{c(t)})_{t\in\mathbb{R}}) = \{T_g \mid g \in H_3(\mathbb{R})\}$. Then*

(i) *T has MSJ and*
(ii) *the actions $(T_g)_{g\in H_{2,a}}$ and $(T_g)_{g\in H_{2,b}}$ have MSJ.*

Proof. (i) Let $\nu \in J_2^e(T)$. It follows from Theorem 4.1(i) that there is a probability measure κ on $H_3(\mathbb{R})$ and a non-negative real $\theta \leq 1$ such that

$$\nu = \theta\mu \times \mu + (1-\theta)\int_{H_3(\mathbb{R})} \mu_{T_g}\, d\kappa(g).$$

Since ν is $(T\times T)$-ergodic and $\mu\times\mu \in J_2(T)$, either $\theta = 0$ or $\theta = 1$. We consider the former case. We note that ν is $(T\times T)$-invariant if and only if κ is invariant under all inner automorphisms of $H_3(\mathbb{R})$. Since κ is finite, it is supported on the center of $H_3(\mathbb{R})$. The ergodicity of ν implies that κ is a singleton. Therefore $\nu = \mu_{T_{c(t)}}$ for some $t \in \mathbb{R}$. Thus, T has MSJ$_2$. Now Theorem 4.1(i) implies that T has MSJ.

(ii) is shown in a similar way. □

5. Mixing Poisson and mixing Gaussian actions of Heisenberg group

We first construct a mixing infinite measure preserving rank-one action of $H_3(\mathbb{R})$. Let $(F_n, C_{n+1})_{n=0}^{\infty}$ be sequence of subsets in $H_3(\mathbb{R})$ satisfying (I)–(IV) from Section 2. Suppose, in addition, that (2-2) is not satisfied. Let T be the (C, F)-action associated with $(F_n, C_{n+1})_{n=0}^{\infty}$. Let $(X, \mathfrak{B}, \mu)$ denote the space of this action. Then T is of rank one along $(F_n)_{n=1}^{\infty}$ and $\mu(X) = \infty$.

Theorem 5.1. *Suppose that the following conditions are satisfied:*

(i) $F_nF_n^{-1}F_nC_n \subset F_{n+1}$,
(ii) *the sets $F_nc_1c_2^{-1}F_n^{-1}$, $c_1 \neq c_2 \in C_{n+1}$, and $F_nF_n^{-1}$ are all pairwise disjoint and*
(iii) $\#C_n \to \infty$ *as* $n \to \infty$.

Then T is mixing.

Proof. Let $(g_n)_{n=1}^\infty$ be a sequence in $H_3(\mathbb{R})$ such that $g_n \to \infty$. We verify that this sequence (or, a subsequence of it) is mixing, i.e. $\mu(T_{g_n} D_1 \cap D_2) \to 0$ for all subsets $D_1, D_2 \subset X$ of finite measure. Without loss of generality we may assume that $g_n \in F_n F_n^{-1} \setminus F_{n-1}F_{n-1}^{-1}$. Take $A, B \subset F_n$. It follows from (i) and (2-3)–(2-6) that

$$\begin{aligned}\mu(T_{g_n}[A]_n \cap [B]_n) &= \mu([g_n AC_{n+1} \cap BC_{n+1}]_{n+1}) \\ &= \sum_{c_1, c_2 \in C_{n+1}} \mu([g_n Ac_1 \cap Bc_2]_{n+1}) \\ &= \mu([g_n A \cap B]_n) + \sum_{c_1 \neq c_2 \in C_{n+1}} \mu([g_n Ac_1 \cap Bc_2]_{n+1})\end{aligned}$$

If $\mu([g_n Ac_1 \cap Bc_2]_{n+1}) > 0$ for some $c_1 \neq c_2 \in C_{n+1}$ then $g_n \in F_n c_2 c_1^{-1} F_n^{-1}$. Since $g_n \in F_n F_n^{-1}$, we obtain a contradiction with (ii). Hence

$$\mu(T_{g_n}[A]_n \cap [B]_n) = \mu([g_n A \cap B]_n).$$

Suppose now that $A = A^* C_{n-1}$, $B = B^* C_{n-1}$ for some subsets $A^*, B^* \subset F_{n-1}$. Then

$$\mu([g_n A \cap B]_n) = \sum_{c_1, c_2 \in C_n} \mu([g_n A^* c_1 \cap B^* c_2]_n).$$

It follows from (ii) that there is no more than one pair $(c_1, c_2) \in C_n \times C_n$ such that $c_1 \neq c_2$ and $\mu([g_n A^* c_1 \cap B^* c_2]_n) > 0$. Hence

$$\mu([g_n A \cap B]_n) = \mu([g_n A^* \cap B^*]_{n-1}) + \mu([g_n A^* c_1 \cap B^* c_2]_n).$$

If $\mu([g_n A^* \cap B^*]_{n-1}) > 0$ then $g_n \in F_{n-1} F_{n-1}^{-1}$, a contradiction. On the other hand,

$$\mu([g_n A^* c_1 \cap B^* c_2]_n) \leq \mu([g_n A^* c_1]_n) = \mu([A^*]_{n-1})/\#C_n$$

by (2-6). Therefore

$$\mu(T_{g_n}[A]_n \cap [B]_n) \leq \mu([A^*]_{n-1})/\#C_n = \mu([A]_n)/\#C_n.$$

It remains to use the standard approximation of D_1 and D_2 with cylinders $[A]_n$ and $[B]_n$ respectively and apply (iii). $\square$

We now recall that given a σ-finite infinite non-atomic standard measure space $(X, \mathfrak{B}, \mu)$, there is a canonical way to associate s standard probability space $(\widetilde{X}, \widetilde{B}, \widetilde{\mu})$ which is called the *Poisson suspension* of $(X, \mathfrak{B}, \mu)$ (see [CFS], [Ro]). Moreover, there exists a continuous homomorphism from the group $\mathrm{Aut}(X, \mu)$ of μ-preserving transformations of X to the group $\mathrm{Aut}(\widetilde{X}, \widetilde{\mu})$ of $\widetilde{\mu}$-preserving transformations of $\widetilde{X}$. The image of a transformation $S \in \mathrm{Aut}(X, \mu)$ under this homomorphism is denoted by $\widetilde{S}$. It is called the *Poisson suspension* of S. Given a μ-preserving action T of $H_3(\mathbb{R})$, we consider a $\widetilde{\mu}$-preserving action $\widetilde{T} = (\widetilde{T}_g)_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ and call it the *Poisson suspension* of T. Consider the Koopman representation U_T of $H_3(\mathbb{R})$ in $L^2(X, \mu)$. Then we can associate to U_T a (probability preserving) Gaussian action $T^* = (T_g^*)_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ (see e.g. [Gl]). The Koopman representations $U_{\widetilde{T}}$ and U_{T^*} of $H_3(\mathbb{R})$ are unitarily equivalent.

Corollary 5.2. *Let T be an action of $H_3(\mathbb{R})$ constructed in Theorem 5.1. Then the Poisson suspension $\widetilde{T}$ of T is mixing of all orders. The corresponding Gaussian action T^* of $H_3(\mathbb{R})$ is also mixing of all orders.*

Proof. The first assertion follows from [Ro, Theorem 4.8]. Since $U_{\widetilde{T}}$ and U_{T^*} are unitarily equivalent and the property of mixing is spectral (i.e. it is a property of the associated Koopman representation), we obtain that T^* is also mixing. The multiple mixing of T^* follows now from [Le]. $\square$

6. Asymmetry in actions of Heisenberg group

In this section we construct a (C, F)-action T of $H_3(\mathbb{R})$ with ergodic flow $(T_{c(t)})_{t\in\mathbb{R}}$ such that the transformation $T_{c(1)}$ is not conjugate to $T^{-1}_{c(1)}$.

Suppose that n is divisible by 3 and on the n-th step of the inductive construction of the sequence $(C_{n+1}, F_n)_{n=0}^{\infty}$ we have defined $F_n = I(\alpha_n, \alpha_n, \gamma_n)$ for some parameters α_n and β_n. We are going to define C_{n+1} and F_{n+1}. Let $\phi_n := \phi_{\alpha_n,\alpha_n,\gamma_n}$. We set

$$s_n(j) := \begin{cases} c(0) & \text{if } j \equiv 0 \pmod 5, \\ c(1) & \text{if } j \equiv 1 \pmod 5 \text{ or } j \equiv 2 \pmod 5, \\ c(2) & \text{if } j \equiv 3 \pmod 5 \text{ or } j \equiv 4 \pmod 5, \end{cases}$$

$c_{n+1}(j) := s_n(j)\phi_n(0, 0, j)$ and $C_{n+1} := \{c_{n+1}(j) \mid |j| \le n\}$. Next, let $F_{n+1} := I(\alpha_n, \alpha_n, \gamma_{n+1})$, where γ_{n+1} is the minimal positive real such that $F_nC_{n+1} \subset I(\alpha_n, \alpha_n, \gamma_{n+1})$.

If n is not divisible by 3, we do the n-th construction step exactly as in Section 3.

In such a way we define completely the sequence $(C_{n+1}, F_n)_{n=0}^{\infty}$. The conditions (I)–(IV) and (2-2) from Section 2 are all satisfied. Denote by T the associated (C, F)-action of $H_3(\mathbb{R})$. The probability space of this action is denoted by (X, μ).

Theorem 6.1. *The flow $(T_{c(t)})_{t\in\mathbb{R}}$ is weakly mixing and rigid. It is not conjugate to the flow $(T^{-1}_{c(t)})_{t\in\mathbb{R}}$. If $\gamma_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then the transformation $T_{c(1)}$ is not conjugate to $T^{-1}_{c(1)}$.*

Proof. The fact that $(T_{c(t)})_{t\in\mathbb{R}}$ is weakly mixing follows immediately from the definition of C_{n+1}, F_{n+1} when $3 \nmid n$ and the proof of Lemma 3.4 (see also Corollary 3.5). It follows from the definition of C_{n+1} when $3 \mid n$ that this flow is rigid. Indeed, it is easy to verify that $T_{\phi_n(0,0,5)} \to \mathrm{Id}$ as $n \to \infty$ and $3 \mid n$.

We now prove the second claim. Fix n which is divisible by 3. Take subsets $A, B, C, D \subset X$. Partition C_{n+1} into C^j_{n+1}, $j = 0, \dots, 4$, as follows

$$C^j_{n+1} := \bigsqcup_{t\equiv j \pmod 5} F_n c_{n+1}(t)$$

and set $F_{n+1}^j := F_n C_{n+1}^j$. Let $l_n := c(1)\phi_n(0,0,1)$. We now claim that

$$\begin{aligned}
&\mu(F_{n+1}^0 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to \frac{1}{5}\mu(A \cap T_{c(1)}^{-1} B \cap T_{c(1)}^{-2} C \cap T_{c(1)}^{-2} D),\\
&\mu(F_{n+1}^1 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to \frac{1}{5}\mu(A \cap T_{c(1)} B \cap C \cap T_{c(1)}^{-1} D),\\
&\mu(F_{n+1}^2 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to \frac{1}{5}\mu(A \cap B \cap T_{c(1)} C \cap D),\\
&\mu(F_{n+1}^3 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to \frac{1}{5}\mu(A \cap B \cap C \cap T_{c(1)} D),\\
&\mu(F_{n+1}^4 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to \frac{1}{5}\mu(A \cap T_{c(1)}^{-1} B \cap T_{c(1)}^{-1} C \cap T_{c(1)}^{-1} D)
\end{aligned}$$

as $n \to \infty$ with $n \in 3\mathbb{N}$. We verify only the first (from the top) claim. It is straightforward that

$$\begin{aligned}
&\frac{\#(l_n C_{n+1}^0 \triangle c(1) C_{n+1}^1)}{\# C_{n+1}^0} \to 1, \quad \frac{\#(l_n C_{n+1}^1 \triangle C_{n+1}^2)}{\# C_{n+1}^1} \to 1,\\
&\frac{\#(l_n C_{n+1}^2 \triangle C_{n+1}^3)}{\# C_{n+1}^2} \to 1, \quad \frac{\#(l_n C_{n+1}^3 \triangle c(1)^{-1} C_{n+1}^4)}{\# C_{n+1}^3} \to 1,\\
&\frac{\#(l_n C_{n+1}^4 \triangle c(1)^{-1} C_{n+1}^0)}{\# C_{n+1}^4} \to 1.
\end{aligned} \tag{6-1}$$

Take some subsets $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \subset F_n \cap c(2) F_n$. It follows from (6-1) that

$$\begin{aligned}
&\mu(F_{n+1}^0 \cap [\widetilde{A}]_n \cap T_{l_n} [\widetilde{B}]_n \cap T_{l_n^2} [\widetilde{C}]_n \cap T_{l_n^3} [\widetilde{D}]_n)\\
=&\mu([\widetilde{A} C_{n+1}^0]_{n+1} \cap T_{l_n} [\widetilde{B} C_{n+1}^4]_{n+1} \cap T_{l_n^2} [\widetilde{C} C_{n+1}^3]_{n+1} \cap T_{l_n^3} [\widetilde{D} C_{n+1}^2]_{n+1}) + \overline{o}(1)\\
=&\mu([\widetilde{A} C_{n+1}^0]_{n+1} \cap [c(1)^{-1} \widetilde{B} C_{n+1}^0]_{n+1} \cap [c(1)^{-2} \widetilde{C} C_{n+1}^0]_{n+1} \cap [c(1)^{-2} \widetilde{D} C_{n+1}^0]_{n+1}) + \overline{o}(1)\\
=&\mu([(\widetilde{A} \cap c(1)^{-1} \widetilde{B} \cap c(1)^{-2} \widetilde{C} \cap c(1)^{-2} \widetilde{D}) C_{n+1}^0]_{n+1}) + \overline{o}(1)\\
=&\frac{1}{5}\mu([\widetilde{A} \cap c(1)^{-1} \widetilde{B} \cap c(1)^{-2} \widetilde{C} \cap c(1)^{-2} \widetilde{D}]_n) + \overline{o}(1)\\
=&\frac{1}{5}\mu([\widetilde{A}]_n \cap T_{c(1)}^{-1} [\widetilde{B}]_n \cap T_{c(1)}^{-2} [\widetilde{C}]_n \cap T_{c(1)}^{-2} [\widetilde{D}]_n) + \overline{o}(1).
\end{aligned}$$

Hence approximating A, B, C, D with cylinders $[\widetilde{A}]_n, [\widetilde{B}]_n, [\widetilde{C}]_n, [\widetilde{D}]_n$ respectively and passing to the limit we deduce that $\mu(F_{n+1}^0 \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) \to 0.2\mu(A \cap T_{c(1)}^{-1} B \cap T_{c(1)}^{-2} C \cap T_{c(1)}^{-2} D)$, as claimed.

We now obtain that

$$\begin{aligned}
\lim_n 5\mu(A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D) &= \lim_n 5 \sum_{j=0}^{4} \mu(F_{n+1}^j \cap A \cap T_{l_n} B \cap T_{l_n^2} C \cap T_{l_n^3} D)\\
&= \mu(A \cap T_{c(1)}^{-1} B \cap T_{c(1)}^{-2} C \cap T_{c(1)}^{-2} D)\\
&+ \mu(A \cap T_{c(1)} B \cap C \cap T_{c(1)}^{-1} D) + \mu(A \cap B \cap T_{c(1)} C \cap D)\\
&+ \mu(A \cap B \cap C \cap T_{c(1)} D) + \mu(T_{c(1)} A \cap B \cap C \cap D),
\end{aligned}$$

where the limit is taken along the sequence of n divisible by 3. In particular, substituting $A = B = D$ and $C = X$ we obtain

$$\lim_n 5\mu(A \cap T_{l_n} A \cap T_{l_n^3} A) \geq \mu(A) \tag{6-2}$$

for each subset $A \subset X$. On the other hand, take a subset $A \subset X$ such that $\mu(A \cap T_{c(1)} A) = \mu(A \cap T_{c(1)}^2 A) = 0$. Then substituting $B = X$ and $A = C = D$, we obtain

$$\lim_n 5\mu(A \cap T_{l_n}^{-1} A \cap T_{l_n}^{-3} A) = \lim_n 5\mu(A \cap T_{l_n^2} A \cap T_{l_n^3} A) = 0. \tag{6-3}$$

It follows from (6-2) and (6-3) that the flows $(T_{c(t)})_{t\in\mathbb{R}}$ and $(T_{c(-t)})_{t\in\mathbb{R}}$ are not conjugate. Thus the second claim of the theorem is proved.

If all γ_n are integers, then l_n is power of $c(1)$ and the third claim of the theorem follows from the second one. $\square$

Remark 6.2. In a similar way we can construct *infinite* measure preserving rank-one actions T of $H_3(\mathbb{R})$ for which the claims of Theorem 6.1 hold.

7. Spectral analysis for actions of Heisenberg group

Let T be an action of $H_3(\mathbb{R})$ on a standard probability space $(X, \mathfrak{B}, \mu)$. Denote by U the corresponding Koopman representation of $H_3(\mathbb{R})$ in $L^2(X, \mu)$. Consider a spectral decomposition of U (we refer to Section 1 for the notation):

$$L^2(X,\mu) = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l_U^{1,2}(\alpha,\beta)} \mathbb{C}\, d\sigma_U^{1,2}(\alpha,\beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l_U^3(\gamma)} L^2(\mathbb{R}, \lambda_{\mathbb{R}})\, d\sigma_U^3(\gamma) \quad \text{and}$$

$$U(g) = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l_U^{1,2}(\alpha,\beta)} \pi_{\alpha,\beta}(g)\, d\sigma_U^{1,2}(\alpha,\beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l_U^3(\gamma)} \pi_\gamma(g)\, d\sigma_U^3(\gamma).$$

We assume that the measures $\sigma_U^{1,2}$ and σ_U^3 are finite. Our purpose in this section is to write some easy (almost straightforward) but important corollaries from the spectral decomposition of U.

Given $\alpha \in \mathbb{R}$, we denote by χ_α the continuous character $\chi_\alpha(t) := e^{i\alpha t}$ of $\mathbb{R}$. We now compute the restriction of U to the subgroup $H_{2,a}$. For that we identify $H_{2,a}$ with $\mathbb{R}^2$ via the mapping $c(t_3)a(t_1) \mapsto (t_3, t_1)$. It is easy to see that $\pi_{\alpha,\beta} \restriction H_{2,a} = \chi_0 \otimes \chi_\alpha$. The restriction of π_γ to $H_{2,a}$ is the tensor product of χ_γ and the left regular representation of $\mathbb{R}$. Therefore the 'projection' of σ_U^3 to $H_{2,a} = \mathbb{R}^2$ is equivalent to the product $\sigma_U^3 \times \lambda_{\mathbb{R}}$. Hence we obtain the following spectral decomposition for $U \restriction H_{2,a}$:

$$L^2(X,\mu) = \int_{\mathbb{R}}^{\oplus} \bigoplus_{j=1}^{l_U^1(\alpha)} \mathbb{C}\, d\sigma_U^1(\alpha) \oplus \int_{\mathbb{R}^*\times\mathbb{R}}^{\oplus} \bigoplus_{j=1}^{l_U^3(\gamma)} \mathbb{C}\, d\sigma_U^3(\gamma) d\lambda_{\mathbb{R}}(\alpha) \quad \text{and}$$

$$U(c(t_3)a(t_1)) = \int_{\mathbb{R}}^{\oplus} \chi_\alpha(t_1) I_\alpha\, d\sigma_U^1(\alpha) \oplus \int_{\mathbb{R}^*\times\mathbb{R}}^{\oplus} \chi_\gamma(t_3)\chi_\alpha(t_1) I_\gamma\, d\sigma_U^3(\gamma) d\lambda_{\mathbb{R}}(\alpha),$$

where σ_U^1 is the projection of $\sigma_U^{1,2}$ under the map $\mathbb{R}^2 \ni (\alpha,\beta) \mapsto \alpha \in \mathbb{R}$, $l_U^1(\alpha)$ is the integral of $l_U^{1,2}$ by the conditional measure $\sigma_U^{1,2}|_{\{\alpha\}\times\mathbb{R}}$ and I_α and I_γ are identity operators in $\bigoplus_{j=1}^{l_U^1(\alpha)} \mathbb{C}$ and $\bigoplus_{j=1}^{l_U^3(\gamma)} \mathbb{C}$ respectively.

Proposition 7.1.

(i) *The maximal spectral type of* $U \restriction H_{2,a}$ *contains the measure* $\delta_0 \times \sigma_U^1 + \sigma_U^3 \times \lambda_{\mathbb{R}}$ *on* $\mathbb{R}^2$. *The corresponding spectral multiplicity map is*

$$\mathbb{R}^2 \ni (\gamma, \alpha) \mapsto \begin{cases} l_U^1(\alpha) & \text{if } \gamma = 0, \\ l_U^3(\gamma) & \text{otherwise.} \end{cases}$$

(ii) *The maximal spectral type of* $U \restriction \{c(t) \mid t \in \mathbb{R}\}$ *contains the measure* $\sigma_U^{1,2}(\mathbb{R}^2)\delta_0 + \sigma_U^3$. *The corresponding spectral multiplicity map is*

$$\mathbb{R} \ni \gamma \mapsto \begin{cases} \int l_U^{1,2}\, d\sigma^{1,2} & \text{if } \gamma = 0, \\ \infty & \text{otherwise.} \end{cases}$$

(iii) *If* $(T_{c(t)})_{t\in\mathbb{R}}$ *is ergodic then* $\sigma_U^{1,2}(\mathbb{R}^2 \setminus \{0,0\}) = 0$, *i.e. there are no non-trivial one-dimensional representations in the spectral decomposition of* U. *The maximal spectral type of* T *equals the maximal spectral type of the restriction* T *to the center of* $H_3(\mathbb{R})$ *(modulo the natural identification).*

(iv) *If* $(T_{c(t)})_{t\in\mathbb{R}}$ *is ergodic then* T *has a simple spectrum if and only if* $U \restriction H_{2,a}$ *has a simple spectrum.*

(v) *If* $(T_{c(t)})_{t\in\mathbb{R}}$ *is ergodic then* $(T_{a(t)})_{t\in\mathbb{R}}$ *has Lebesgue (restricted) maximal spectral type.*

(vi) *If* $(T_{c(t)})_{t\in\mathbb{R}}$ *is mixing then* $(T_g)_{g\in H_{2,a}}$ *is mixing.*

We note that (vi) is weaker than Theorem 3.7. However we include it here because it follows almost directly from the spectral decomposition for U, i.e. it is simpler (shorter) than the proof of Theorem 3.7.

Corollary 7.2. *Let* T *and* S *be two actions of* $H_3(\mathbb{R})$ *such that the flows* $(T_{c(t)})_{t\in\mathbb{R}}$ *and* $(S_{c(t)})_{t\in\mathbb{R}}$ *are ergodic. Then the set of spectral multiplicities of the Cartesian product* $T \times S = (T_g \times S_g)_{g\in H_3(\mathbb{R})}$ *contains infinity.*

Idea of the proof. The claim follows from Propositions 3.10, 7.1(iii) and the following well know fact:

$$\pi_\gamma \otimes \pi_{\gamma'} = \begin{cases} \bigoplus_{j=1}^{\infty} \pi_{\gamma+\gamma'} & \text{if } \gamma + \gamma' \neq 0, \\ \int_{\mathbb{R}^2} \pi_{\alpha,\beta}\, d\lambda_{\mathbb{R}}(\alpha)\, d\lambda_{\mathbb{R}}(\beta) & \text{if } \gamma + \gamma' = 0. \end{cases}$$

□

It follows, in particular, that the set of spectral multiplicities of each (non-degenerated) Gaussian and Poisson action of $H_3(\mathbb{R})$ contains infinity. The 'non-degenerated' here means that that the corresponding (reduced) maximal spectral types are non-atomic.

8. Concluding remarks and open problems

(1) Actions T and S of $H_3(\mathbb{R})$ on probability spaces (X, μ) and (Y, ν) respectively are called *disjoint in sense of Furstenberg* [Fu] if $\mu \times \nu$ is the only $(T_g \times S_g)_{g\in H_3(\mathbb{R})}$-invariant measure on $X \times Y$ with marginals μ and ν on X and Y respectively. Modifying the construction from Section 3 one can

obtain an uncountable family of mutually disjoint mixing rank-one actions of $H_3(\mathbb{R})$ with MSJ.

(2) The examples of mixing rank-one actions of $H_3(\mathbb{R})$ in Section 3 are of stochastic nature. The choice of the 'spacer' maps s_n is not explicit (as in the Ornstein's example of mixing $\mathbb{Z}$-action [Or]). Is it possible to construct explicit, concrete examples of such actions by analogy with the classical $\mathbb{Z}$-staircases [Ad]?

(3) Are there smooth (differentiable) models for mixing rank-one actions of $H_3(\mathbb{R})$?

(4) All the results of this paper obtained for the 3-dimensional Heisenberg group extend naturally to the Hisenberg groups $H_{2k+1}(\mathbb{R})$, $k > 1$, of higher dimensions and some 'generalized' Heisenberg groups [Ki]. Is it possible to extend them to the class of all (or a sufficiently wide subclass of) simply connected nilpotent Lie groups?

(5) We conjecture that a mixing rank-one action of $H_3(\mathbb{R})$ is mixing of all orders.

(6) Whether each 2-fold simple weakly mixing action of $H_3(\mathbb{R})$ is simple?

(7) Let an action T of $H_3(\mathbb{R})$ have MSJ. Does this imply that the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is simple? Is this true in the particular case when T is of rank one?

(8) Can we construct an action T of $H_3(\mathbb{R})$ such that the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is ergodic and conjugate to its inverse but the flows $(T_{a(t)})_{t\in\mathbb{R}}$ and $(T_{b(t)})_{t\in\mathbb{R}}$ are non-conjugate or even disjoint in the sense of Furstenberg? We note that T is always 'spectrally' symmetric, i.e. the Koopman representation U_T is unitarily equivalent to $U_T \circ \theta$, where θ denotes the flip in $H_3(\mathbb{R})$. Indeed, $\pi_\gamma \circ \theta$ is unitarily equivalent to $\pi_{-\gamma}$ for each $\gamma \in \mathbb{R}^*$ and the measure σ^3_U is quasi-invariant under the inversion $\mathbb{R}^* \ni t \mapsto -t \in \mathbb{R}^*$ in view of Proposition 7.1(iii). In particular, we now deduce from Proposition 7.2(v) $(T_{a(t)})_{t\in\mathbb{R}}$ and $(T_{b(t)})_{t\in\mathbb{R}}$ have the same Lebesgue maximal spectral type.

(9) Is there an action T of $H_3(\mathbb{R})$ such that the flow $(T_{c(t)})_{t\in\mathbb{R}}$ is ergodic and conjugate to its inverse, the flows $(T_{a(t)})_{t\in\mathbb{R}}$ and $(T_{b(t)})_{t\in\mathbb{R}}$ are conjugate but T is asymmetric, i.e. it is not conjugate to the action $(T_{\theta(g)})_{g\in H_3(\mathbb{R})}$?

(10) Suppose that T and S are two disjoint ergodic actions of $H_3(\mathbb{R})$. Does this imply that the flows $(T_{c(t)})_{t\in\mathbb{R}}$ and $(S_{c(t)})_{t\in\mathbb{R}}$ are also disjoint?

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